# Hidden symmetry and Collective behavior

Bin Ao, Zhigang Zhu, Liang Huang, and Lei Yang[1]
*Institute of Modern Physics, Chinese Academy of Science, Lanzhou 730000, China*

(Dated: April 23, 2013)

We study the relationship between the partially synchronous state and the coupling structure in general dynamical systems. Our results show that, on the contrary to the widely accepted concept, topological symmetry in a coupling structure is the sufficient condition but not the necessary condition. Furthermore, we find the necessary and sufficient condition for the existence of the partial synchronization and develop a method to obtain all of the existing partially synchronous solutions for all nonspecific dynamics from a very large number of possible candidates.

PACS numbers: 05.45.Xt

Synchronization has attracted extensive attention in the physical, biological, ecological, and other systems [2]. The theory of synchronization focuses on the dynamical behavior of many-body coupled systems. Of the types of synchronization, global synchronization (GS) is a well researched topic [3]. Partial synchronization (PaS), which is the synchronization only emerges in part of a system, is a more general synchronous phenomenon. And PaS has been observed in the systems whose parameters are outside the GS regime. A “simple” case, the PaS in a globally coupled system, is studied in detail in Refs. [4, 6]. And random and other complex coupling structures are studied in Refs. [5–7]. However, the underlying mechanism of PaS is yet far from clear. A fundamental difficulty lies in finding the PaS solutions for a given structure. And the problem remains open.

Considering the dynamics [10]

$$\mathbf{X}_{n+1} = \mathbf{F}(\mathbf{X}_n) + \varepsilon(C \otimes \Gamma)\mathbf{F}(\mathbf{X}_n), \tag{1}$$

where $\mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N)'$ represents the state of a system consisted of $N$ subsystems $\{\mathbf{x}^i\}_{i=1}^N$ and ”$()'$” means the transpose of a matrix. The independent state, $\mathbf{x}^1 \neq \mathbf{x}^2 \neq \cdots \neq \mathbf{x}^N$, and the GS state, $\mathbf{x}^1 = \mathbf{x}^2 = \cdots = \mathbf{x}^N$, universally exist for common couplings [11]. The PaS solution, however, may not exist in a given coupled system. As an example, for the structure shown in Fig. 1(a), all 203 candidate PaS solutions (e.g., $\mathbf{x}^1 = \mathbf{x}^2$ and $\mathbf{x}^3 \neq \mathbf{x}^4 \neq \mathbf{x}^5 \neq \mathbf{x}^6$; this is a kind of exhaust algorithm) do not satisfy Eq. (1).

In recent years, it has been developed to a common belief that there exist close relation between the topological symmetry of a coupling structure and the PaS state. For example, an asymmetric PaS pattern that does not follow a symmetrical structure has never been observed [5], the theory of symmetric group can be used to describe the periodic PaS state in several regular structures with the same symmetry [8], and all PaS states corresponding to each topological symmetry in a ring have been observed [9]. Based on these examples, one could suppose that symmetry is the necessary and sufficient condition for the existence of the PaS state.

When the number of subsystems is small, the above statement may seem to hold true. The coupling structure in the inset of Fig. 1(b) given by the adjacency matrix $A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$. Here, nodes 1 and 2 are symmetric, and $A_4$ is invariant under the permutations $1 \to 2$ and $2 \to 1$. The curves $d_{1,2}(\varepsilon)$ and $d_{2,3}(\varepsilon)$ [12] are shown in Fig. 1 (b), where $\varepsilon$ is the coupling strength. The synchronous solution $\mathbf{x}^1 = \mathbf{x}^2 \neq \mathbf{x}^3 \neq \mathbf{x}^4$ is observed in the region $\varepsilon \in [0.3, 0.45] \cup [0.9, 1]$. Thus, the PaS state will be achieved with the corresponding symmetry in $A_4$ among the symmetrical nodes.

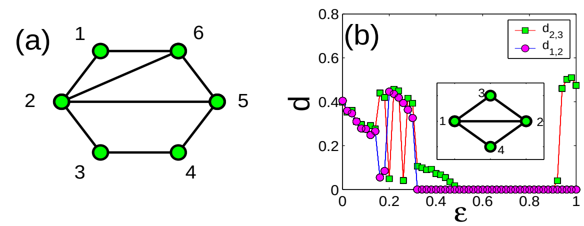


FIG. 1: (a) A topological structure of the coupled system without any partial synchronous solution. (b) The average distance $d_{1,2}$ and $d_{2,3}$ versus the coupling strength $\varepsilon$ with the structure shown in the inset graph.

A more complex case is shown in Fig. 2 (a) with the same dynamics as above. There are two clusters and their nodes are denoted $1, 2, \cdots, n_1$ and $n_1+1, n_1+2, \cdots, n_1+n_2$. Node $i$ $(1 \leq i \leq n_1)$ is coupled to nodes $i-k, i-k+1, \cdots, i+k$, $n_1+i-l, n_1+i-l+1, \cdots, n_1+i+l$; and node $n_1+i$ $(1 \leq i \leq n_1)$ is coupled to the nodes $n_1+i-k$, $n_1+i-k+1, \cdots$, $n_1+i+k$, $i-l, i-l+1, \cdots, i+l$. Obviously, there is ”rotational” symmetry in every cluster; in other words, the adjacency matrix is invariant under a ”rotation” permutation in each cluster; that is, $1 \to 2$, $2 \to 3$, $\cdots$, $n_1-1 \to n_1$, $n_1 \to 1$, and $n_1+1 \to n_1+2$, $n_1+2 \to n_1+3$, $\cdots$, $2n_1-1 \to 2n_1$, $2n_1 \to n_1+1$. We define a $M \times M$ matrix $R_M$, where $(R_M)_{M,1} = 1$, $(R_M)_{i,i+1} = 1$ $(i = 1, 2, \cdots, M-1)$, and the other elements are 0. Thus, the permutation matrix [13] of this transformation will be $T_d = R_{n_1} \oplus R_{n_2}$, where $\oplus$ represents the direct sum of two matrices. Fig. 2 (b) shows the time-averaged variation in all subsystems $\sigma(\varepsilon)$, and that in the two clusters, $\sigma_1(\varepsilon)$ and $\sigma_2(\varepsilon)$ [12] for $n_1 =$

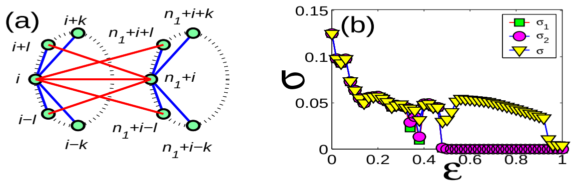


FIG. 2: (a) The scheme of a topological structure with the "rotational" symmetry. (b) The variation of the two clusters $\sigma_1$, $\sigma_2$ and the whole system $\sigma$ as functions of $\varepsilon$ with $n_1 = n_2 = 100$, $k = 40$, and $l = 10$.

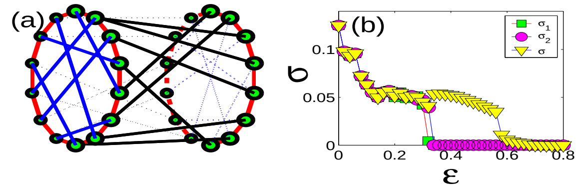


FIG. 3: (a) The scheme of an equal degree random structure. (b) The variance of the two clusters $\sigma_1$, $\sigma_2$ and the whole network $\sigma$ as functions of the coupling strength $\varepsilon$ in the parameter $n_1 = n_2 = 100$, $p = 1$, and $p_r = 0.5$.

$n_2 = 100$, $k = 40$, and $l = 10$. For $\varepsilon \in [0.45, 1]$, $\sigma_1 = \sigma_2 = 0$; that is, $\mathbf{x}_1 = \cdots = \mathbf{x}_{n_1}$ and $\mathbf{x}_{n_1+1} = \cdots = \mathbf{x}_N$. The PaS solution of the "rotational" symmetrical nodes is observed. The above examples show that the presence of symmetry in a structure may be necessary and sufficient for the existence of PaS solutions.

In this Letter, we investigate in detail the relationship between the PaS solution and the coupling structure. The PaS solution is defined as follows: For a dynamical system with phase space $\mathbb{R}^{Nm}$, a $K$-cluster synchronous solution is a $Km$-dimensional subspace $V$ of $\mathbb{R}^{Nm}$. It can be represented by

$$\begin{array}{c} \mathbf{x}^{i_1^1} = \mathbf{x}^{i_1^2} = \cdots = \mathbf{x}^{i_1^{n_1}}, \\ \mathbf{x}^{i_2^1} = \mathbf{x}^{i_2^2} = \cdots = \mathbf{x}^{i_2^{n_2}}, \\ \cdots\cdots \\ \mathbf{x}^{i_K^1} = \mathbf{x}^{i_K^2} = \cdots = \mathbf{x}^{i_K^{n_K}}, \end{array} \tag{2}$$

where $\mathbf{x}^{i_a^b}$ denotes the $b$th subsystem in the $a$th cluster; and $\{n_i\}_{i=1}^K$ are the sizes of each cluster, where $\sum_{i=1}^K n_i = N$ $(N > K > 1)$. The GS and independent solutions are the particular cases where $K = 1$ and $K = N$, respectively. The relationship between the PaS solution and the coupling structure could be described by the following two questions:

Question A: If one finds symmetry in a coupling structure, can a corresponding PaS solution be obtained?

Eq. (2) can also be described by its matrix form:

$$(T \otimes I_m)\mathbf{X} = \mathbf{X}, \forall \mathbf{X} \in V, \tag{3}$$

where $T$ is a permutation matrix and $I_m$ is a $m$-dimension identity matrix. That is, $\mathbf{X} \in V$ is invariant under the permutation transformation $T \otimes I_m$, so $V$ is the invariant subspace of $T \otimes I_m$ and the eigen-subspace of $T \otimes I_m$ corresponding to eigenvalue 1. That $V$ is the invariant subspace of the dynamical system in Eq. (1) requires

$$(C \otimes \Gamma)\mathbf{X} \in V, \quad \forall \mathbf{X} \in V. \tag{4}$$

Next, if there is symmetry $T$ in structure $C$, then $C$ will be invariant under the permutation transformation $T$. The mathematical representation is

$$T^{-1}CT = C, \tag{5}$$

or the matrices $C$ and $T$ are commutative in multiplication.

Question A can be represented by the following mathematical statement:

$$If\ Eq.\ (3)\ and\ Eq.\ (5)\ hold,\ then\ Eq.\ (4)\ holds.$$

Multiplying the two sides of Eq. (5) by $\mathbf{X} \in V$ and arbitrary $\Gamma$, we have

$$[(TC) \otimes \Gamma]\mathbf{X} = [(CT) \otimes \Gamma]\mathbf{X}, \forall \mathbf{X} \in V. \tag{6}$$

Combining Eq. (3) with Eq. (6) gives

$$(T \otimes I_m)(C \otimes \Gamma)\mathbf{X} = (C \otimes \Gamma)\mathbf{X}, \forall \mathbf{X} \in V. \tag{7}$$

Thus, $(C \otimes \Gamma)\mathbf{X}$ is also the eigenvector of $T \otimes I_m$ with eigenvalue 1. Then, Eq. (4) will be satisfied for all $\mathbf{X} \in V$. We conclude that for a symmetrical structure, the dynamical system has a corresponding PaS solution.

Question B: If one finds a PaS solution, can the corresponding symmetry in the coupling structure be observed?

Here, an interesting example is shown in Fig. 3 (a). Let us consider two clusters; each has $n$ subsystems and every subsystem is randomly connected to $[pn] + 1$ subsystems in the same cluster ($p$ is a probability and $[pn]$ means the integer part of $pn$) and $[p_r n] + 1$ subsystems in the other cluster ($p_r$ is also a probability). All the subsystems are of equal degree and the connections between two clusters are also of equal degree. Fig. 3 (b) shows the variance of the two clusters ($\sigma_1$ and $\sigma_2$) and of the whole system ($\sigma$) as functions of the coupling strength $\varepsilon$ for $n = 100$, $p = 1$, and $p_r = 0.5$. As $\varepsilon \in [0.34, 0.76]$, $\sigma_1 = \sigma_2 = 0$, and $\sigma > 0$, the PaS solution is observed. Due to the random connections between the subsystems, there is no symmetry in the structure.

Question B can be represented by the following mathematical statement:

$$If\ Eq.\ (3)\ and\ Eq.\ (4)\ hold,\ then\ Eq.\ (5)\ holds.$$

The following relations could be derived:

$$\begin{array}{rc} & (C \otimes \Gamma)\mathbf{X} \in V, (T \otimes I_m)\mathbf{X} = \mathbf{X}, \forall \mathbf{X} \in V, \\ \Longleftrightarrow & (T \otimes I_m)(C \otimes \Gamma)\mathbf{X} = (C \otimes \Gamma)\mathbf{X}, \\ \Longleftrightarrow & (T \otimes I_m)(C \otimes \Gamma)\mathbf{X} = (C \otimes \Gamma)(T \otimes I_m)\mathbf{X}, \\ \Longleftrightarrow & [(TC) \otimes (I_m\Gamma)]\mathbf{X} = [(CT) \otimes (\Gamma I_m)]\mathbf{X}, \\ \Longleftrightarrow & [(TC - CT) \otimes \Gamma]\mathbf{X} = 0. \end{array} \tag{8}$$

Obviously, Eq. (8) is not equivalent to Eq. (5). We can conclude that it is possible for a PaS solution to exist in a dynamical system without any symmetry; Fig. 3 presents an example. Another important point is that, in fact, the necessary and sufficient condition can be drawn from Eq. (8) itself.

For a simple representation but not loss its generality, we only discuss the case $m = 1$. The component form of Eq. (8) is

$$\sum_{j=1}^{N}(C_{kj} - C_{ij})\mathbf{x}^j = 0, \forall \mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N) \in V. \quad (9)$$

for $T_{ik} = 1$. By relabeling the subsystems to group subsystems of the same cluster together, we rewrite Eq. (2) as below: For cluster $s$,

$$\mathbf{x}^{N_s+1} = \mathbf{x}^{N_s+2} = \cdots = \mathbf{x}^{N_s+n_s} \equiv \mathbf{y}^s, \ (s = 1, 2, \cdots, K), \quad (10)$$

where $N_s = 0$ when $s = 1$, and $N_s = \sum_{i=1}^{s-1} n_i$ when $s = 2, \cdots, K$ [14]. Then Eq. (9) can be grouped into $K$ terms as

$$\sum_{s=1}^{K}[\sum_{j=N_s+1}^{N_s+n_s}(C_{ij} - C_{kj})]\mathbf{y}^s = 0, \ (i = 1, 2, \cdots, N). \quad (11)$$

This is the $i$th row of Eq. (8). For all nonspecific dynamics, we always have

$$\sum_{j=1}^{n_s}(C_{i,N_s+j} - C_{k,N_s+j}) = 0, (s = 1, 2, \cdots, K) \quad (12)$$

for $T_{ik} = 1 (i = 1, 2, \cdots, N)$.

For a subsystem $i$ which belongs to cluster $s$, we define the degree of subsystem $i$, which is contributed by the $s'$th cluster ($s' \neq s$), as the **external degree** of $i$ from $s'$; And the degree of subsystem $i$, which is contributed by the $s$th cluster as the **internal degree** of $i$. Since $C_{ij}$ may be a fraction or zero, the external and internal degrees of a subsystem can also be fraction or zero.

For subsystem $i$ in cluster $s$, $\sum_{j=1}^{n_{s'}} C_{i,N_{s'}+j}$ is the external degrees of $i$ from $s'$. And because of $T_{ik} = 1$, both subsystems $i$ and $k$ are in cluster $s$. Then from Eq. (12), the external degrees of $i$ and $k$ from $s'$ should be the same. And, $\sum_{j=1}^{n_s} C_{i,N_s+j} - C_{ii}$ is the internal degrees of $i$. Because of $C_{kk} = C_{ii} = -1$, the rest part of Eq. (12) shows that the internal degrees of $i$ and $k$ should be the same.

Considering the above two different situations in Eq. (12), we can have the following necessary and sufficient conditions of the existence of PaS solutions: ***S1***: **the external degrees of every subsystems in a cluster from another cluster should be the same.** ***S2***: **the internal degrees of the subsystems in a cluster should also be the same**.

The two conditions form a complete representation of Eq. (8). Now we have a clear physical picture of the existence of a PaS solution. And from the two conditions, we can derive all the PaS solutions in a given structure using the following strategy: The key is to divide the structure into several substructures, each comprising subsystems that can synchronize with each other but not with subsystems in other substructures. Then the PaS solutions of the whole system will be combinations of all the possible solutions of every substructure.

To satisfy **S1** and **S2**, the PaS cannot be achieved between directly connected subsystems whose degrees are coprime [15]. So an $N \times N$ matrix $S$ can be defined such that $S_{im} = 1$ if the degrees of the subsystems $i$ and $m$ are not coprime or $C_{im} = 0$, and $S_{im} = 0$ for other cases. Thus, the system can be divided into several groups of subsystems, and if the subsystems $i$ and $m$ are in different groups, then $S_{im} = 0$. We can then suppose that $\mathbf{x}^i = \mathbf{x}^m$ when $S_{im} = 1$. Thus, an equation like Eq.(9) can be represented as $\sum_{j_+ \in J_+} C^+_{j_+}\mathbf{x}^{j_+} + \sum_{j_- \in J_-} C^-_{j_-}\mathbf{x}^{j_-} = 0$, where $j_+ \in J_+ = \{j | C^+_j \equiv C_{mj} - C_{ij} > 0\}$ and $j_- \in J_- = \{j | C^-_j \equiv C_{mj} - C_{ij} < 0\}$. Positive (negative) coefficients $C^+_{j_+}$ ($C^-_{j_-}$) indicate that the internal or external degrees of subsystem $m$ from subsystem $j_+$ ($j_-$) is greater (less) than that of the subsystem $i$ from $j_+$ ($j_-$). Therefore, according to the requirement of **S1** and **S2**, subsystem $j_+$ should belong to a cluster that includes one or more subsystems in $J_-$ so that the positive degree difference $C^+_{j_+}$ can be counteracted. But subsystem $j_+$ may not be synchronous with some members of $J_-$ (some elements $S_{j_+j_-}$, $\forall j_- \in J_-$ may be 0); if $C^+_{j_+} + \sum_{j_- \in J_-} S_{j_+j_-}C^-_{j_-} > 0, \exists j_+ \in J_+$, then Eq.(9) cannot be satisfied. This means $\mathbf{x}^i = \mathbf{x}^m$ is impossible and $S_{im}$ should be reset to 0. For the reasons specified above, if $C^-_{j_-} + \sum_{j_+ \in J_+} S_{j_-j_+}C^+_{j_+} < 0, \exists j_- \in J_-$, $S_{im}$ should be reset to 0. To let $i, m = 1, \cdots, N$ and perform the operations discussed above, we obtain a new $S$. Thus, the system can be divided into some subgroups by considering the new $S$. This procedure can be repeated until the new $S$ equals the old $S$, and the size of the subgroups of the subsystems will decrease with every repetition. After repeated subdivision, we obtain a number of final groups, which can be grouped into clusters using the exhaust algorithm. If S1 can be satisfied for every pair of clusters and S2 can be satisfied in each individual cluster, combinations of these clusters are PaS solutions of the whole system.

The number of candidate PaS solutions for any structure rapidly increases with N [17]. Thus, for a random structure with $N = 1000$ and $p = 0.995$ [16], it would be impossible to test all the candidates. But we can use the above procedure to find all PaS solutions. Three equal-degree groups that include at least two subsystems are found: *group* $1, \{172, 261\}$; *group* $2, \{532, 910\}$; and *group* $3, \{231, 277, 503, 555, 756\}$. Thus, there are $2 \times 2 \times 52 = 208$ different PaS solutions. Fig. 4(a) shows the variances of the whole system, $\sigma$, and the three clusters, $\sigma_1$, $\sigma_2$, and $\sigma_3$, as functions of the coupling strength $\varepsilon$. It is easy to find the PaS solutions $x_{172} = x_{261}$, $x_{532} = x_{910}$, and $x_{231} = x_{277} = x_{503} = x_{555} = x_{756}$.

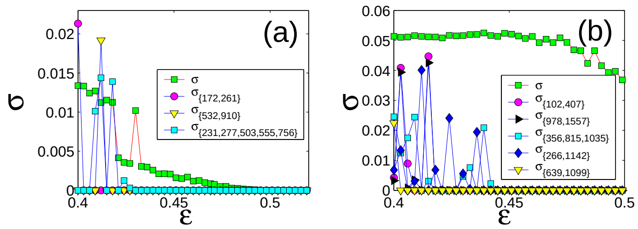


FIG. 4: The variance of the PaS clusters and the whole system $\sigma$ as functions of the coupling strength $\varepsilon$ (a) A random structure: $N = 1000$, $p = 0.995$. (b) A Barabási-Albert scale free structure: $N = 1600$.

For the case of a Barabási-Albert scale free structure ($N = 1600$) [16, 18], Fig. 4(b) show the existence of PaS clusters $\{102, 407\}$, $\{978, 1557\}$, $\{356, 815, 1035\}$, $\{266, 1142\}$, and $\{639, 1099\}$ [16].

In conclusion, we studied the relationship between the coupling structure and the PaS state in general dynamical systems. The necessary and sufficient condition of the existence of a PaS state was found from an exact proof. And the results are counterintuitive; the existence of a PaS state does not require symmetry, as assumed previously. And all of the candidate PaS solutions exist in a globally coupled system. Furthermore, as the exhaust algorithm cannot be used to obtain all of the existent PaS solutions, we developed a method to find all these solutions for a given structure. We focused mainly on the existence of PaS solutions in this letter. But to determine the stability of these solutions, the conditional Lyapunov exponent should be calculated by regarding the PaS manifold as the condition. Note that the proof also can also be applied to differential dynamical systems and the conclusions are the same.

We are grateful for discussions with Ping Lin, Jinfang Zhang, Ying Hu and Wei Kang. This work was supported in part by the 100 Person Project of the Chinese Academy of Sciences.

---

[1] Corresponding author: lyang@impcas.ac.cn

[2] Y. Tanguy *et al.*, Phys. Rev. Lett. **96**, 053902 (2006); A. Vilfan, F. Julicher, *ibid.* **96**, 058102 (2006); I. Fischer *et al.*, *ibid.* **97**, 123902 (2006); E. Emberly *et al.*, *ibid.* **96**, 038303 (2006); D. Battogtokh *et al.*, *ibid.* **96**,148102 (2006); Y-C. Lai *et al.*, *ibid.* **98**, 108102 (2007); R. E. Amritkar *et al.*, *ibid.* **96**, 258102 (2006).

[3] L. M. Pecora *et al.*, Phys. Rev. Lett. **80**, 2109 (1998); L. Huang *et al.*, *ibid.* **97**, 164101 (2006); J. Z. Yang *et al.*, *ibid.* **80**, 496 (1998).

[4] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989).

[5] S. C. Manrubia *et al.*, Phys. Rev. E **60**, 1579 (1999).

[6] R. E. Amritkar *et al.*, Phys. Rev. E **72**, 016212 (2005).

[7] D. H. Zanette *et al.*, Phys. Rev. E **57**, 276 (1998); O. Popovych *et al.*, Physica D **168-169**, 106 (2002); S. Sinha, Phys. Rev. E **66**, 016209 (2002); J. Garcia-Ojalvo *et al.*, Phys. Rev. Lett. **98**, 034101 (2007); M. Rosenblum *et al.*, Phys. Rev. Lett. **98**, 064101 (2007); H. Nakao *et al.*, Phys. Rev. Lett. **98**, 184101 (2007).

[8] Y. Jiang *et al.*, Phys. Rev. E **68**, 065201(R) (2003).

[9] Y. Zhang *et al.*, Phys. Rev. E **63**, 026211 (2001).

[10] This is one of the widely used forms of the coupling nonlinear maps. One can use a certain $m$-dimensional map $\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n)$, $\mathbf{x} = (x_n^1, x_n^2, \cdots, x_n^m)$ as a subsystem to construct a coupled system $\mathbf{X} = (\mathbf{x}^1, \mathbf{x}^2, \cdots, \mathbf{x}^N)'$, $\mathbf{F} = (\mathbf{f}^1, \mathbf{f}^2, \cdots, \mathbf{f}^N)'$. The common interest is linear coupling between the subsytems; thus, the coupling term will be $\varepsilon C \otimes \Gamma \mathbf{F}(\mathbf{X})$, where $\varepsilon$ the coupling strength. $\Gamma : \mathbb{R}^m \to \mathbb{R}^m$ characterizes the coupling scheme among the subsystems and $C$ is the Laplacian matrix. $C_{ij} = A_{ij}/k_i$ for $j \neq i$ and $C_{ii} = -1$, where $k_i$ is the degree of subsystem $i$ and $A_{ij}$ is an element of the adjacency matrix $A$ of the coupling structure. To gain further insights in numerical simulation, we set $f(x)$ as be the logistic map $f(x) = 4x(1-x)$, but the proof does not depend on the dynamics.

[11] A. Pikovsky *et al.*, *Synchronization: A universal concept in nonlinear sciences* (Cambridge University Press, 2001); A. Arenas *et al.*, Phys. Rep. **469**, 93 (2008).

[12] One can define the average distance between two oscillators $i$ and $j$ by $d_{ij} = \lim_{N\to\infty}(N^{-1}\sum_{n=1}^{N} \|\mathbf{x}_n^i - \mathbf{x}_n^j\|)$ and the time-averaged variation among subsystems $\sigma = \lim_{N'\to\infty} \frac{1}{N'} \sum_{n=1}^{N'} \sqrt{\sum_{i=1}^{N} (\mathbf{x}_n^i - \frac{1}{N}\sum_{i=1}^{N} \mathbf{x}_n^i)^2}$, and state that they are in a synchronous state in the sense that $d_{ij} < 10^{-12}$.

[13] G. B. Dantzig *et al.*, *Linear Programming* (Springer-Verlag, 1997), p. 327; L. Smith, *Linear Algebra* (Springer, 1998), p. 156.

[14] The general form of $T$ will be $T = \bigoplus_{i=1}^{K} R_{n_i}$ ($i = 1, 2, \cdots, K$). Then the general form of $\mathbf{X}$ will be $\mathbf{X} = (\mathbf{y}^1 \otimes 1_{1,n_1}, \mathbf{y}^2 \otimes 1_{1,n_2}, \cdots, \mathbf{y}^K \otimes 1_{1,n_K})'$ where $1_{M,N}$ is an $M \times N$ matrix in which every element is 1. Then the $T_d$ in the second example is only a special case of this form, and the permutation transformation done to $A_4$ can also be represented by $T_4 = R_2 \oplus 1 \oplus 1$.

[15] Since $C_{ij}$ in Eq.(12) can be a fraction, subsystems whose degrees are not coprime with each other may satisfy the requirement of this eqution (e.g., $1/6 + 1/6 + 1/6 = 1/4 + 1/4$). On the other hand, according to Eq. (12), a cluster with zero internal degree is also allowed. To satisfy the requirement of **S2**, subsystems in such a cluster cannot be connected with each other directly. Thus, two directly connected subsystems cannot be included in the same such cluster.

[16] The adjacency matrices of the two structures and a comlete set of the PaS solutions of the second structres can be found in http://cluster.impcas.ac.cn/data.html

[17] For example, there are only 52 possible solutions (including the GS and the independent solutions) in a structure with five subsystems, but the number will be more than $1.38 \times 10^9$ when the system size is just 15.

[18] A. -L. Barabási, R. Albert, Science **286**, 509 (1999).